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Investigation of a Griffith crack subject to anti-plane shear by using the non-local theory

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Abstract

Field equations of the non-local elasticity are solved to determine the state of stress in a plate with a Griffith crack subject to the anti-plane shear. Then a set of dual-integral equations is solved using Schmidt's method. Contrary to the classical elasticity solution, it is found that no stress singularity is present at the crack tip. The significance of this result is that the fracture criteria are unified at both the macroscopic and the microscopic scales.

1. Introduction

In several previous papers (see e.g. Eringen et al., 1977a; Eringen, 1978, 1979), Eringen discussed the state of stress near the tip of a sharp line crack in an elastic plate subject to uniform tension, shear and anti-plane shear. The field equations employed in the solution of these problems are those of the theory of non-local elasticity. The solutions obtained did not contain any stress singularity, thus resolving a fundamental problem that persisted over many years. This enables us to employ the maximum stress hypothesis to deal with fracture problems in a natural way, and also the non-local elasticity has a big potential to understand the behavior of composite materials. However, Eringen's solution is not exact (see e.g. Eringen, 1979).

In the present paper, the same problem which was treated by Eringen (see e.g. Eringen, 1979) is reworked using somewhat different approach. Fourier transform is applied and a mixed boundary value problem is reduced to a set of dual-integral equations. In solving the dual-integral equations, the crack surface displacement is expanded in a series using Jacobi's polynomials and Schmidt's method (see e.g. Morse and Feshbach, 1958) is used. This process is quite different from that adopted in the paper of Eringen's (see e.g. Eringen, 1979). The solution in this paper is more exact and more reasonable than Eringen's. As expected, it does not contain the stress singularity at the

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crack tip, thus clearly indicating the physical nature of the problem, namely, in the vicinity of the geometrical discontinuities in the body, the non-local intermolecular forces are dominant. For such problems, therefore, one must resort to theories incorporating non-local effects, at least in the neighborhood of the discontinuities. The stress along the crack's line depends on the crack length.

2. Basic equations of non-local elasticity

Basic equations of linear, homogeneous, isotropic, non-local elastic solids, with vanishing body force are (see e.g. Eringen, 1979)

$$\tau_{kl,k} = 0 \tag{1}$$

$$\tau_{kl} = \int_{V'} [\lambda'(|X'-X|)e_{rr}(X')\delta_{kl} + 2\mu'(|X'-X|)e_{kl}(X')]\,\mathrm{d}V' \tag{2}$$

$$e_{kl} = \tfrac{1}{2}(u_{k,l} + u_{l,k}) \tag{3}$$

where the only difference from classical elasticity is in the stress constitutive eqns (2) in which the stress $\tau_{kl}(X)$ at a point X depends on the strains $e_{kl}(X')$, at all points of the body. For homogeneous and isotropic solids there exist only two material constants, $\lambda'(|X'-X|)$ and $\mu'(|X'-X|)$ which are functions of the distance $|X'-X|$. The integral in (2) is over the volume V of the body enclosed within a surface ∂V. In this paper we employ Cartesian coordinates x_k with the usual convention that a free index takes the values (1, 2, 3), and repeated indices are summed over the range (1, 2, 3). Indices following a comma represent partial differentiation, e.g.

$$u_{k,l} = \partial u_k / \partial x_j$$

In the papers (see e.g. Eringen, 1974, 1977b), it can be obtained in the form of $\lambda'(|X'-X|)$ and $\mu'(|X'-X|)$ for which the dispersion curves of plane elastic waves coincide with those known in lattice dynamics. Among several possible curves the following has been found to be very useful

$$(\lambda', \mu') = (\lambda, \mu)\alpha(|X'-X|) \tag{4}$$

$$\alpha(|X'-X|) = \alpha_0 \exp[-(\beta/a)^2(X'-X)(X'-X)] \tag{5}$$

where β is a constant, a is the lattice parameter. λ and μ are the Lamé constants of classical elasticity. α_0 is determined by the normalization

$$\int_V \alpha(|X'-X|)\,\mathrm{d}V(X') = 1 \tag{6}$$

In the present work, the non-local elastic moduli was given by (4) and (5). Substituting (5) into (6) it can be obtained, in two dimensional space,

$$\alpha_0 = \frac{1}{\pi}(\beta/a)^2 \tag{7}$$

Substitution of eqns (4)–(5) into eqn (2) yields

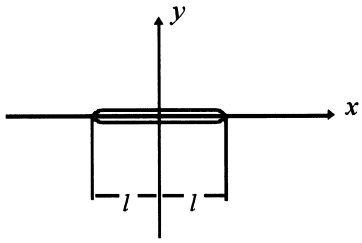

Fig. 1. Line crack subject to anti-plane shear.

$$\tau_{kl}(X) = \int_V \alpha(|X'-X|)\sigma_{kl}(X')\,\mathrm{d}V(X') \tag{8}$$

where

$$\begin{aligned}\sigma_{ij}(X') &= \lambda e_{rr}(X')\delta_{ij} + 2\mu e_{ij}(X') \\ &= \lambda u_{r,r}(X')\delta_{ij} + \mu[u_{i,j}(X') + u_{j,i}(X')]\end{aligned} \tag{9}$$

The expression (9) is the classical Hook's law.

3. The crack model

Consider an elastic plate in the $(x_1 = x, x_2 = y)$ plane weakened by a line crack of length $2l$ along the x-axis. The plate is subjected to a constant anti-plane shear stress $\tau_{yz} = \tau_0$ along the surface of the crack (see Fig. 1). For this problem, it has

$$\begin{aligned}&u_1 = u_2 = 0, \quad u_3 = w(x, y) \\ &\sigma_{xz} = \mu\frac{\partial w}{\partial x}, \quad \sigma_{yz} = \mu\frac{\partial w}{\partial y}, \quad \text{all other } \sigma_{kl} = 0\end{aligned} \tag{10}$$

The boundary conditions are (see e.g. Eringen, 1979):

$$w(x, 0) = 0, \quad \text{for } |x| > l \tag{11}$$

$$\tau_{yz}(x, 0) = \tau_0, \quad \text{for } |x| \leqslant l \tag{12}$$

$$w(x, y) = 0 \quad \text{as } (x^2 + y^2)^{1/2} \to \infty \tag{13}$$

Substituting eqn (9) into eqn (1), using Green–Gauss theorem and equation (10), it can be obtained (see e.g. Eringen, 1979):

$$\mu\int_V\int \alpha(|x'-x|, |y'-y|)\nabla'^2 w(x', y')\,\mathrm{d}x'\,\mathrm{d}y' - \int_{-l}^{l} \alpha(|x'-x|, |y|)[\sigma_{yz}(x', 0)]\,\mathrm{d}x' = 0 \tag{14}$$

where the boldface bracket indicates a jump at the crack line.

Under the applied anti-plane shear load on the unopened surfaces of the crack, the displacement field possesses the following symmetry regulations

$$w(x,-y) = -w(x,y)$$

Using this in (9) we find that

$$[\sigma_{yz}(x',0)] = 0 \tag{15}$$

Hence the line integral in (14) vanishes. By taking the Fourier transform of (14) with respect to x', it can be shown that the general solution of (14) is identical to that of

$$\frac{\mathrm{d}^2\bar{w}(s,y)}{\mathrm{d}y^2} - s^2\bar{w}(s,y) = 0 \tag{16}$$

almost everywhere. Here a superposed bar indicates the Fourier transform, e.g.

$$\bar{f}(s,y) = \int_0^\infty f(x,y)\exp(isx)\,\mathrm{d}x$$

The general solution of eqn (16) ($y \geqslant 0$) satisfying (13) is:

$$w = \frac{2}{\pi}\int_0^\infty A(s)\,\mathrm{e}^{-sy}\cos(xs)\,\mathrm{d}s \tag{17}$$

where $A(s)$ is to be determined from the boundary conditions.

The stress field, according to (8), is given by

$$\tau_{yz} = -\frac{2\mu}{\pi}\int_0^\infty A(s)s\,\mathrm{d}s\int_0^\infty \mathrm{d}y'\int_{-\infty}^\infty [\alpha(|x'-x|,|y'-y|) + \alpha(|x'-x|,|y'+y|)]\,\mathrm{e}^{-sy}\cos(sx')\,\mathrm{d}x' \tag{18}$$

Substituting for α from (5), according to the reference (see e.g. Eringen, 1979) and the boundary conditions (11) and (12), it can be obtained

$$\int_0^\infty sA(s)\mathrm{erfc}(\varepsilon s)\cos(sx)\,\mathrm{d}s = -\frac{\pi\tau_0}{2\mu}, \quad 0 \leqslant x \leqslant l \tag{19}$$

$$\int_0^\infty A(s)\cos(sx)\,\mathrm{d}s = 0, \quad x > l \tag{20}$$

where

$$\varepsilon = \frac{a}{2\beta}, \quad \mathrm{erfc}(z) = 1-\Phi(z), \quad \Phi(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-t^2)\,\mathrm{d}t \tag{21}$$

Since the only difference between the classical and non-local equations is in the introduction of the function $\mathrm{erfc}(\varepsilon s)$, it is logical to utilize the classical solution to convert the system (19) and (20) to an integral equation of the second kind which is generally better behaved. For $a = 0$, then

$\mathrm{erfc}(\varepsilon s) = 1$ and eqns (19) and (20) reduce to the dual integral equations for same problem in classical elasticity. To determine the unknown function $A(s)$, the dual-integral eqns (19) and (20) must be solved.

4. Solution of the dual-integral equation

The dual eqns (19) and (20) can not be transformed into the second Fredholm integral equation (see e.g. Eringen, 1979), because the kernel of the second kind Fredholm integral equation in the paper of Eringen (1979) is divergent. The kernel of the second kind Fredholm integral equation in Eringen's paper (1979) can be written as follows:

$$L(x,u) = (xu)^{1/2} \int_0^\infty tk(\varepsilon t) J_0(xt) J_0(ut)\,\mathrm{d}t, \quad 0 \leqslant x,u \leqslant 1 \tag{22}$$

where $J_n(x)$ is the Bessel function of order n.

$$k(\varepsilon t) = -\Phi(\varepsilon t), \quad \Phi(z) = 2\pi^{1/2} \int_0^z \exp(-t^2)\,\mathrm{d}t \tag{23}$$

$$\lim_{t\to\infty} k(\varepsilon t) \neq 0 \quad \text{for } \varepsilon = \frac{a}{2\beta l} \neq 0 \tag{24}$$

$$J_0(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{1}{4}\pi\right) \quad \text{for } x \gg 0 \tag{25}$$

The limit of $tk(\varepsilon t)J_0(xt)J_0(ut)$ does not equal to zero for $t \to \infty$. So the kernel $L(x,u)$ in Eringen's paper is divergent (see e.g. Eringen, 1979). Of course, the dual integral equations can be considered to be a single integral equation of the first kind with a discontinuous kernel (see e.g. Eringen, 1977). It is well-known in the literature that integral equations of the first kind are generally ill-posed in sense of Hadamard, i.e. small perturbations of the data can yield arbitrarily large changes in the solution. This makes the numerical solution of such equations quite difficult. For overcoming the difficult, the Schmidt method (see e.g. Morse and Feshbach, 1958) is used to solve the dual-integral eqns (19) and (20). The displacement w was represented by the following series:

$$w(x,0) = \sum_{n=1}^{\infty} a_n P_{2n-2}^{(1/2,1/2)}\left(\frac{x}{l}\right)\left(1 - \frac{x^2}{l^2}\right)^{1/2}, \quad \text{for } 0 \leqslant |x| \leqslant l \tag{26}$$

$$w(x,0) = 0, \quad \text{for } l \leqslant |x| \tag{27}$$

where a_n are unknown coefficients to be determined and $P_n^{(1/2,1/2)}(x)$ is a Jacobi polynomial (see e.g. Erdelyi, 1954). The Fourier cosine transform for eqn (26) is (see e.g. Erdelyi, 1954)

$$A(s) = \bar{w}(s,0) = \sum_{n=1}^{\infty} a_n B_n J_{2n-1}(ls) s^{-1}, \tag{28}$$

where

$$B_n = 2\sqrt{\pi}(-1)^{n-1}\frac{\Gamma\left(2n-\frac{1}{2}\right)}{(2n-2)!} \tag{29}$$

where $\Gamma(x)$ and $J_n(x)$ are the Gamma and Bessel functions, respectively.

Substitution of eqn (28) into eqns (19) and (20), respectively, eqn (20) has been automatically satisfied by using the Fourier transform. Then the remaining eqn (19) reduces to the form for $x \leqslant l$

$$\sum_{n=1}^{\infty} a_n B_n \int_0^{\infty} \operatorname{erfc}(\varepsilon s) J_{2n-1}(ls)\cos(sx)\,\mathrm{d}s = -\frac{\pi\tau_0}{2\mu} \tag{30}$$

For a large s, the integrands of eqn (30) almost decrease exponentially. So the semi-infinite integral in eqn (30) can be evaluated numerically by Filon's method (see e.g. Amemiya and Taguchi, 1969). Thus eqn (19) can be solved for coefficients a_n by the Schmidt method (see e.g. Morse and Feshbach, 1958). For brevity, eqn (30) can be rewritten as

$$\sum_{n=1}^{\infty} a_n E_n(x) = U(x), \quad 0 \leqslant x \leqslant l \tag{31}$$

where $E_n(x)$ and $U(x)$ are known functions and coefficients a_n are unknown and will be determined. A set of functions $P_n(x)$ which satisfy the orthogonality condition

$$\int_0^l P_m(x)P_n(x)\,\mathrm{d}x = N_n\delta_{mn}, \quad N_n = \int_0^l P_n^2(x)\,\mathrm{d}x \tag{32}$$

can be constructed from the function, $E_n(x)$, such that

$$P_n(x) = \sum_{i=1}^{n} \frac{M_{in}}{M_{nn}} E_i(x) \tag{33}$$

where M_{in} is the cofactor of the element d_{in} of D_n, which is defined as

$$D_n = \begin{bmatrix} d_{11}, d_{12}, d_{13}, \ldots, d_{1n} \\ d_{21}, d_{22}, d_{23}, \ldots, d_{2n} \\ d_{31}, d_{32}, d_{33}, \ldots, d_{3n} \\ \cdots \\ \cdots \\ \cdots \\ d_{n1}, d_{n2}, d_{n3}, \ldots, d_{nn} \end{bmatrix}, \quad d_{in} = \int_0^l E_i(x)E_n(x)\,\mathrm{d}x \tag{34}$$

Using eqns (31)–(34), it can be obtained

$$a_n = \sum_{j=n}^{\infty} q_j \frac{M_{nj}}{M_{jj}} \tag{35}$$

with

Table 1
Values of $\sum_{n=1}^{10} a_n E_n(x)/\frac{\pi\tau_0}{2\mu}$ and $U(x)/\frac{\pi\tau_0}{2\mu}$ for $a/2\beta l = 0.0005$

x/l	$\sum_{n=1}^{10} a_n E_n(x)/\frac{\pi\tau_0}{2\mu}$	$U(x)/\frac{\pi\tau_0}{2\mu}$
0.55	−0.100309E+01	−1.0
0.60	−0.100531E+01	−1.0
0.75	−0.995341E+00	−1.0
0.80	−0.100439E+01	−1.0
0.90	−0.996281E+00	−1.0
0.95	−0.102530E+01	−1.0
0.96	−0.102805E+01	−1.0
0.97	−0.102363E+01	−1.0
0.98	−0.998501E+00	−1.0
0.99	−0.869906E+00	−1.0

$$q_j = \frac{1}{N_j}\int_0^l U(x)P_j(x)\,\mathrm{d}x \tag{36}$$

5. Numerical calculations and discussion

For a check of accuracy, the values of

$$\sum_{n=1}^{10} a_n E_n(x)$$

and $U(x)$ are given in Table 1 in the cases of $a/2\beta l = 0.0005$. In Table 2, the values of the coefficients a_n are given for $a/2\beta l = 0.0005$.

From about the results and references (see e.g. Itou, 1978, 1979), it can be seen that the Schmidt method is performed satisfactorily if the first ten terms of infinite series to eqn (31) are retained. The behavior of the maximum stress stays steady with the increasing number of terms in (26). When coefficients a_n are known, the entire stress field is obtainable. However, in fracture mechanics, it is of importance to determine stress τ_{yz} along the crack line. τ_{yz} at $y = 0$ is given as follows:

$$\tau_{yz} = -\frac{2\mu}{\pi}\sum_{n=1}^{\infty} a_n B_n \int_0^{\infty} \mathrm{erfc}(\varepsilon s) J_{2n-1}(ls)\cos(sx)\,\mathrm{d}s \tag{37}$$

For $\varepsilon = 0$ at $x = l$ we have the classical stress singularity. However, so long as $\varepsilon \neq 0$, (37) gave a finite stress all along $y = 0$. At $0 < x < l$, τ_{yx}/τ_0 is very close to unity, and for $x > l$, τ_{yx}/τ_0 possesses finite values diminishing from a maximum value at $x = l$ to zero at $x = \infty$. Since $\varepsilon/l > 1/100$ represents a crack length of less than 10^{-6} cm, and such submicroscopic sizes other serious questions arise regarding the interatomic arrangements and force laws, the solution was not

Table 2
Values of $a_n/\frac{\pi\tau_0}{2\mu}$ for $a/2\beta l = 0.0005$

n	$a_n/\frac{\pi\tau_0}{2\mu}$
1	−0.318698E+00
2	−0.127109E−01
3	0.708155E−02
4	0.174376E−02
5	0.127016E−02
6	−0.132851E−03
7	−0.570583E−04
8	−0.981545E−04
9	−0.106541E−04
10	−0.582841E−05

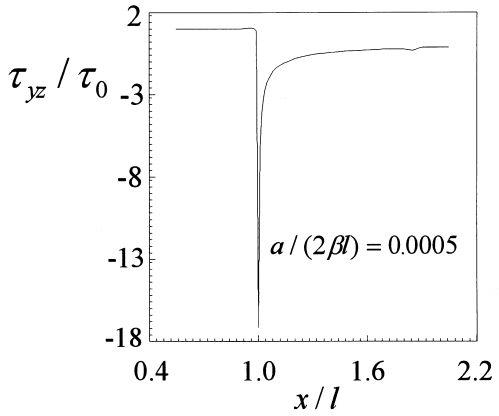

Fig. 2. Anti-plane shear stress.

pursued at such small crack sizes. The stress is computed numerically for Poisson's ratio $v = 0.29$. The semi-infinite numerical integrals, which occur, are evaluated easily by Filon and Simpson's methods (see e.g. Amemiya and Taguchi, 1969) because of the rapid diminution of the integrands. Because the integrands of eqns (31) and (37) are complex, the shear stress along the crack face has a slight variation. The results are plotted in Figs 2–7.

The following observations are made:

(i) The maximum shear stress occurs at the crack tip, and it is finite. Contrary to the classical elasticity solution, it is found that no stress singularity is present at the crack tip.
(ii) The shear stress at the crack tip becomes infinite as the atomic distance $a \to 0$. This is the classical continuum limit of square root singularity.

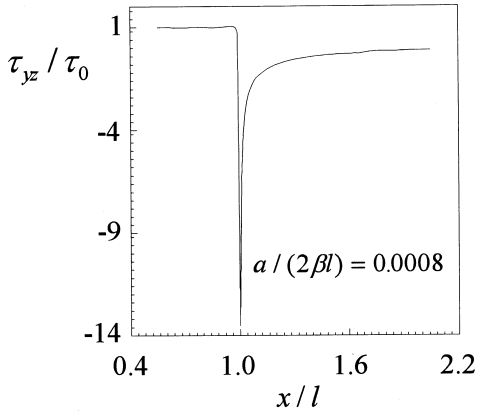

Fig. 3. Anti-plane shear stress.

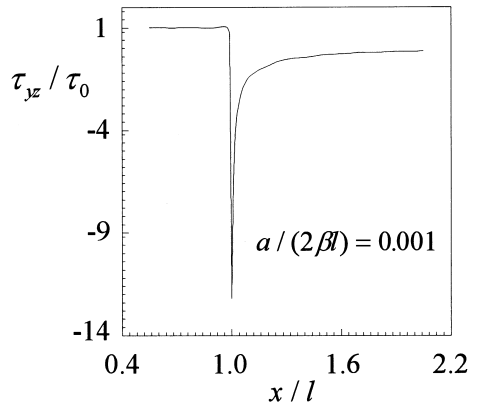

Fig. 4. Anti-plane shear stress.

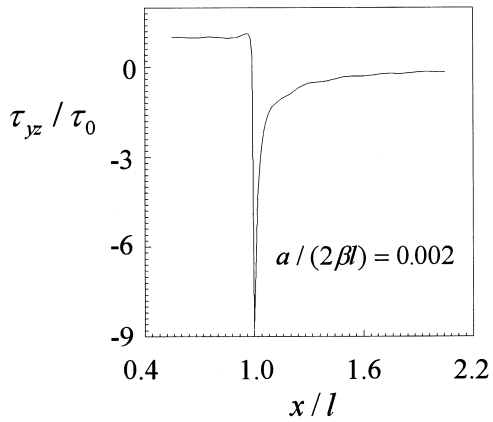

Fig. 5. Anti-plane shear stress.

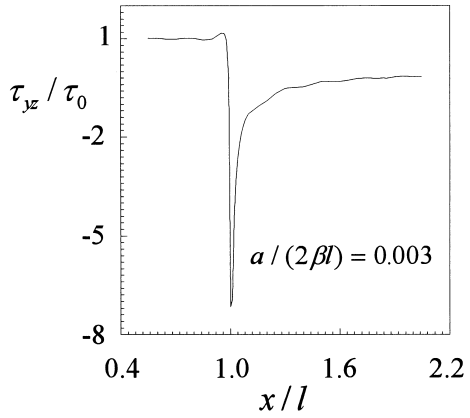

Fig. 6. Anti-plane shear stress.

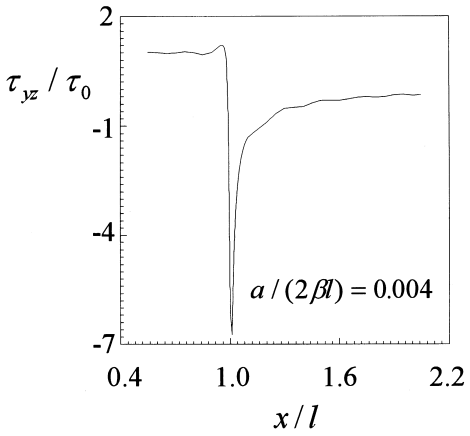

Fig. 7. Anti-plane shear stress.

(iii) For the $a/\beta =$ constant, viz, the atomic distance does not change, the values of the stress concentrations (at the crack tip) becomes higher with the increase of the crack length. Note this fact, experiments indicate that materials with smaller cracks are more resistant to fracture than those with larger cracks (see e.g. Eringen, 1979).

(iv) The significance of this result is that the fracture criteria are unified at both the macroscopic and microscopic scales.

(v) The stress concentration occurs at the crack tip, and this is given by

$$\tau_{yz}(l, 0)/\tau_0 = c_3/\sqrt{a/(2\beta l)}$$

where c_3 converges to $c_3 \approx 0.383$.

(vi) The present results converge to the classical ones when far away from the crack tip.

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